THE PROBLEM OF CONTROL WITH BOUNDED PHASE COORDINATES

PMM Vol. 32, No. 2, 1968, pp. 194-202

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(*Received December 14, 1967*)

The problem of controlling a linear system with bounded phase coordinates is considered. The paper is concerned primarily with the limiting process which leads from the solutions of problems approximating the initial problem to the required solution. The approach employed is based on the interpretation of control problems as moment problems (e.g. see [1] which contains a bibliography of the subject).

1. Formulation of the problem. Let us consider the controlled motion $x(t)$ described by the differential Eq.

$$dx(t) / dt = Ax + Bu + w(t) \tag{1.1}$$

Here x is the n-vector of the phase coordinates; u is the scalar controlling force; $w(t)$ is a continuous n-vector function (the specified disturbance); A and B are constant matrices of the appropriate dimensions.

Problem 1. We are given the time interval $t_0 \leq t \leq T$ and the initial $x(t_0) = x^\circ$ and final $x(T) = x^T$ states of the phase vector x. We are also given m functions $f_k(t)$ ($k = 1,\ldots, m \leq n$) which are continuous on $[t_0, T]$ and strictly positive (for $t > t_0$). We are required to choose from among the forces $u(t)$ which bring system (1.1) from x° to x^T in the time $T - t_0$ under the restrictions

$$|x_k(t)| \leqslant f_k(t) \qquad (t_0 \leqslant t \leqslant T;\ k = 1, \ldots, m) \tag{1.2}$$

a control $u^\circ(t)$ for which

$$\varkappa[u^\circ] = \operatorname{vrai\,max}_t |u^\circ(t)| = \min_u \varkappa[u] = \min_u \operatorname{vrai\,max}_t |u(t)| \tag{1.3}$$

$$(t_0 \leqslant t \leqslant T)$$

We shall call the control $u^\circ(t)$ "optimal".

2. Method of solution and the basic result. Let us partition the interval $t_0 \leq t \leq T$ into N equal parts at the points

$$t_i = t_0 + i\Delta_N t, \qquad \Delta_N t = (T - t_0) / N \qquad (i = 1, \ldots, N)$$

and consider Problem 1, replacing restrictions (1.2) by the conditions

$$|x_k(t_i)| \leqslant f_k(t_i) \qquad (k = 1, \ldots, m;\ i = 1, \ldots, N) \tag{2.1}$$

For brevity we shall refer to this problem as Problem 1_N. We propose to investigate initial Problem 1 by taking the limits ($N \to \infty$) of the solutions of Problem 1_N.

According to the solving procedure of [1], Problem 1_N can be reduced to a moment problem: from among the functions $u_N(t)$ satisfying the relations

$$\int_{t_0}^{T} h_s[T,\tau]\, u_N(\tau)\, d\tau = c_s \qquad (s = 1,\ldots,n) \tag{2.2}$$

$$\int_{t_0}^{T} h_k[t_i,\tau]\, u_N(\tau)\, d\tau - z_{ki} = c_{ki}, \quad |z_{ki}| \leqslant f_k(t_i) \quad (i = 1,\ldots,N-1;\ k = 1,\ldots,m)$$

we are required to find a function $u_N{}^\circ(t)$ such that

$$\varkappa[u^\circ] = \min_u \varkappa[u]$$

Here z_{ki} are constant numbers; $h_j[t,\tau]$ is the j-th component of the vector

$$H[t,\tau] = X[t,\tau]\, B \quad (dX[t,\tau]/dt = AX[t,\tau],\ X[t,t] = E)$$

and $h_j[t,\tau] \equiv 0$ for $\tau \geq t$; the numbers c_k and c_{ki} are, respectively, the k-th components of the vectors

$$c = x^T - X[T,t_0]\, x^\circ - \int_{t_0}^{T} X[T,\tau]\, w(\tau)\, d\tau$$

$$c^{(i)} = -X[t_i,t_0]\, x^\circ - \int_{t_0}^{t_i} X[t_i,\tau]\, w(\tau)\, d\tau \tag{2.3}$$

We assume that system (1.1) is completely controllable [1]. The functions

$$h_s[T,\tau],\ h_k[t_i,\tau] \qquad (s = 1,\ldots,n;\ k = 1,\ldots,m;\ i = 1,\ldots,N-1)$$

are then linearly independent, and problem (2.2), (2.3) is solvable. The solution is provided by the function

$$u_N^\circ(t) = \nu_N^\circ \operatorname{sign} h_N^\circ(t) \tag{2.4}$$

$$h_N^\circ(\tau) = \sum_{s=1}^{n} \lambda_{sN}^\circ h_s[T,\tau] + \sum_{k=1}^{m} \sum_{i=1}^{N-1} l_{kiN}^\circ h_k[t_i,\tau]\, \Delta_N t \tag{2.5}$$

The numbers $\lambda_{sN}{}^\circ$, $l_{kiN}{}^\circ$, $\nu_N{}^\circ$ are the solution of the arbitrary extremum problem

$$\nu_N^\circ = \Phi(\lambda_N^\circ, l_N^\circ) = \max_{\lambda,l} \Phi(\lambda_N, l_N) = \max_{\lambda,l} \frac{S}{J} \tag{2.6}$$

$$S = \sum_{s=1}^{n} \lambda_{sN} c_s + \sum_{k=1}^{m} \sum_{i=1}^{N-1} l_{kiN}\, c_{ki} \Delta_N t - \sum_{k=1}^{m} \sum_{i=1}^{N-1} f_k(t_i)\, |l_{kiN}|\, \Delta_N t$$

$$J = \int_{t_0}^{T} \Big| \sum_{s=1}^{n} \lambda_{sN} h_s[T,\tau] + \sum_{k=1}^{m} \sum_{i=1}^{N-1} l_{kiN} h_k[t_i,\tau]\, \Delta_N t \Big|\, d\tau$$

for

$$\rho^2[\lambda_N, l_N] = \sum_{s=1}^{n} \lambda_{sN}^2 + \sum_{k=1}^{m} \sum_{i=1}^{N-1} l_{kiN}^2 \leqslant 1 \quad \max_i \sum_{k=1}^{m} |l_{kiN}| \leqslant 1 \tag{2.7}$$

We note that by virtue of the above assumptions the denominator in (2.6) differs from zero for all N, and that the number $\nu_N{}^\circ$ is positive.

Let us consider the sequence of partitions of the interval $t_0 \leq t \leq T$ into N equal parts, setting $N = N_\alpha$, $N_\alpha = 2N_{\alpha-1}$ ($\alpha = 1, 2,\ldots$).

We denote by $l_{kN}(t)$ the function (2.8)

$$l_{kN}(t) = l_{kiN} \quad \text{for} \quad t_{i-1} < t \leqslant t_i \quad (i = 1,\ldots,N-1) \quad l_{kN}(t) \equiv 0 \quad \text{for} \quad t_{N-1} < t \leqslant T$$

We can then rewrite relations (2.6) and (2.7) as

$$v_N^\circ = \Phi(\lambda_N^\circ, l_N^\circ(t)) = \max_{\lambda, l} \Phi(\lambda_N, l_N(t)) = \tag{2.9}$$

$$= \max_{\lambda, l(t)} \frac{1}{J^\circ} \{\varphi_1 [\lambda_N, l_N(t)] + o_1(\Delta_N t) - \varphi_3 [l_N(t)] + o_3(\Delta_N t)\}$$

$$J^\circ = \int_{t_0}^{T} |\varphi_2 [\lambda_N, l_N(t); \tau] + o_2(\Delta_N t)| d\tau$$

for

$$\rho^2 [\lambda_N, l_N(t)] = \sum_{s=1}^{n} \lambda_{sN}^2 + \sum_{k=1}^{m} \int_{t_0}^{T} l_{kN}^2(t)\, dt \leqslant 1, \quad \operatorname{vrai\,max}_t \sum_{k=1}^{m} |l_{kN}(t)| \leqslant 1 \tag{2.10}$$

Here

$$\varphi_1 = \sum_{s=1}^{n} \lambda_{sN} c_s + \sum_{k=1}^{m} \int_{t_0}^{T} c_k(t)\, l_{kN}(t)\, dt \tag{2.11}$$

$$\varphi_2 = \sum_{s=1}^{n} \lambda_{sN} h_s [T, \tau] + \sum_{k=1}^{m} \int_{\tau}^{T} l_{kN}(t)\, h_k [t, \tau]\, dt \tag{2.12}$$

$$\varphi_3 = \sum_{k=1}^{m} \int_{t_0}^{T} f_k(t)\, |l_{kN}(t)|\, dt \tag{2.13}$$

The symbols $o_i(\Delta_N t)$ in (2.9) represent quantities which tend to zero as $\Delta_N t \to 0$, and

$$|o_1(\Delta_N t)| = \left| \sum_{k=1}^{m} \int_{t_0}^{T} (c_k(t) - c_{kN}(t))\, l_{kN}(t)\, dt \right| \leqslant k_1 \Delta_N t$$

$$c_{kN}(t) = c_{ki} \qquad \text{for} \quad t_{i-1} < t \leqslant t_i,$$

$$|o_2(\Delta_N t)| = \left| \sum_{k=1}^{m} \int_{\tau}^{T} (h_k [t, \tau] - h_{kN} [t, \tau])\, l_{kN}(t)\, dt \right| \leqslant k_2 \Delta_N t$$

$$h_{kN} = h_{kN} [t_i, \tau] \qquad \text{for} \quad t_{i-1} < t \leqslant t_i$$

$$|o_3(\Delta_N t)| \leqslant \sum_{k=1}^{m} \int_{t_0}^{T} |(f_k(t) - f_{kN}(t))\, l_{kN}(t)|\, dt$$

$$f_{kN}(t) = f_k(t_i) \quad \text{for} \quad t_{i-1} < t \leqslant t_i \qquad (0 < k_j = \text{const} < \infty)$$

where $c_k(t)$-th component of the vector function is

$$c(t) = -X[t, t_0]\, x^\circ - \int_{t_0}^{t} X[t, \tau]\, w(\tau)\, d\tau$$

The ordered system

$$\xi_N^\circ = \{\lambda_N^\circ, l_N^\circ(t)\} = \{\lambda_{1N}^\circ, \ldots, \lambda_{nN}^\circ;\ l_{1N}^\circ(t), \ldots l_{mN}^\circ(t)\}$$

is an element of set (2.10) of the Hilbert space $H\{\xi\}$ with the metric $\rho[\xi] = \rho[\lambda, l(t)]$.

From the property of weak compactness [2] of a sphere in $H\{\xi\}$ we infer that the sequence of quantities ξ_N° contains a weakly convergent sequence $\xi^\circ = \{\lambda^\circ, l^\circ(t)\} = \{\lambda_1^\circ, \ldots, \lambda_n^\circ;$

$l_1^\circ(t),\ldots, l_m^\circ(t)\}$ with a weak limit, where $\rho[\xi^\circ] \le 1$; moreover, by virtue of the second condition of (2.10), vrai $\max_t|l_k^\circ(t)| \le 1$. (We shall retain our symbol $\{\xi_N^\circ\}$ for this subsequence.) We note, further, that the functions $h_s[T, \tau]$, $c_k(t)$ from (2.9) to (2.12) are continuous. The operations ϕ_1 (2.11) and ϕ_2 (2.12) (for a fixed τ) are therefore linear functions over $H\{\xi\}$.

Thus,

$$\lim_{N\to\infty}\varphi_1[\xi_N^*] = \varphi_1[\xi^\circ], \qquad \lim_{N\to\infty}\varphi_2[\xi_N^*;\ \tau] = \varphi_2[\xi^\circ;\ \tau] \tag{2.14}$$

Condition (2.14) for ϕ_2 ensures existence of the limit

$$\lim_{N\to\infty}\int_{t_0}^{T}|\varphi_2[\xi_N^*;\tau] + o_2(\Delta_N t)|\,d\tau = \int_{t_0}^{T}|\varphi_2[\xi^\circ;\tau]|\,d\tau$$

Now let us show that

$$\lim_{N\to\infty}\varphi_3[l_N^*(t)] = \varphi_3[l^\circ(t)] \tag{2.15}$$

We note that the sequence $l_N^\circ(t) = \{l_{1N}^\circ(t),\ldots, l_{mN}^\circ(t)\}$ converges weakly to the quantity $l^\circ(t) = \{l_1^\circ(t),\ldots, l_m^\circ(t)\}$ in the space L_2 of m-vector functions. Recalling that the quantity $\phi_3[l_N(t)]$ is the norm of the element $l_N(t)$, we obtain the inequality [2]

$$\liminf\varphi_3[l_N^*(t)] \geqslant \varphi_3[l^\circ(t)] \quad \text{as} \quad N\to\infty \tag{2.16}$$

We assume that the quantity

$$h^\circ(\tau) = \varphi_2[\xi^\circ;\tau] = \sum_{s=1}^{n}\lambda_s^* h_s[T,\tau] + \sum_{k=1}^{m}\int_{\tau}^{T} l_k^*(t)\,h_k[t,\tau]\,dt \tag{2.17}$$

is not identically equal to zero on a set of zero measure from $[t_0, T]$. It is clear from this that the limit $\liminf \Phi(\lambda_N^\circ, l_N^\circ(t))$ as $N\to\infty$ does, in fact, exist, and that the quantity $\Phi(\lambda^\circ, l^\circ(t))$ has meaning. Let us show that the relation

$$\Phi(\lambda^\circ,\ l^\circ(t)) \leqslant \liminf_{N\to\infty}\Phi(\lambda_N^*,\ l_N^*(t))$$

is valid, thus verifying both the inequality

$$\limsup_{N\to\infty}\varphi_3[l_N^*(t)] \leqslant \varphi_3[l^\circ(t)]$$

and (by virtue of (2.16)) condition (2.15).

Let us assume the contrary. Then

$$\Phi(\lambda^\circ,\ l^\circ(t)) - \liminf_{N\to\infty}\Phi(\lambda_N^*,\ l_N^*(t)) \geqslant \sigma > 0$$

On the basis of the vector function $l^\circ(t)$, which is generally not continuous, we can construct the continuous vector function $l^\circ(t)_\sigma$, each of whose components differs from the corresponding component of the function $l^\circ(t)$ only on some set of measure smaller than σ, and such that

$$|\Phi(\lambda^\circ,\ l^\circ(t)_\sigma) - \Phi(\lambda^\circ,\ l^\circ(t))| \leqslant \sigma/2 \tag{2.18}$$

The latter is possible by virtue of the Luzin theorem [3]. The functions $l_k^\circ(t)_\sigma$ are bounded: $\max_t|l_k^\circ(t)| \le 1$. For the functions $l_k^\circ(t)_\sigma$ we have the relations

$$\int_{t_0}^{T} c_k(t)\,l_k^*(t)_\sigma\,dt = \lim_{N\to\infty}\sum_{i=1}^{N} c_k(t_i)\,l_k^*(t_i)_\sigma$$

$$\int_{\tau}^{T} h_k[t,\tau]\,l_k^*(t)_\sigma\,dt = \lim_{N\to\infty}\sum_{i=i(\tau)}^{N} h_k[t_i,\tau]\,l_k^*(t_i)_\sigma$$

$$\int_{t_0}^{T} f_k(t)\,|\,l_k^{\circ}(t)_{\sigma}\,|\,dt = \lim_{N\to\infty} \sum_{i=1}^{N} f_k(t_i)\,|\,l_k^{\circ}(t_i)_{\sigma}\,|$$

The sums in the right sides of (2.18) are the integral Riemann sums corresponding to the continuous vector function $l_k{}^{\circ}(t)_{\sigma}$. Substituting into the functional $\Phi(\lambda_N, l_N)$ from (2.6), on the one hand, the quantities

$$\lambda^{\circ} = \{\lambda_1^{\circ}, \ldots, \lambda_n^{\circ}\},\quad l^{\circ}(t_i)_{\sigma} = \{l_1^{\circ}(t_i)_{\sigma}, \ldots, l_m^{\circ}(t_i)_{\sigma}\} \quad (i = 1, \ldots, N)$$

and, on the other, the solution

$$\lambda_N{}^{\circ} = \{\lambda_{1N}^{\circ}, \ldots, \lambda_{nN}^{\circ}\},\ \{l_{kiN}^{\circ};\ k = 1, \ldots, m;\ i = 1, \ldots, N-1\}$$

of problem (2.6), (2.7), we obtain the inequality

$$\Phi(\lambda^{\circ},\ l^{\circ}(t_i)_{\sigma}) \leqslant \Phi(\lambda_N^{\circ},\ l_N^{\circ}) \tag{2.19}$$

Let us take the limit $(N \to \infty)$ in both sides of the above inequality. We note that the quantity $\Phi(\lambda_N, l_N{}^{\circ})$ in this inequality can be represented (with allowance for (2.8)) in the form $\Phi(\lambda_N, l_N{}^{\circ}(t))$ (see (2.9) to (2.13)). Then, recalling the weak convergence of $\{\lambda_N{}^{\circ}, l_N{}^{\circ}(t)\}$ to $\{\lambda^{\circ}, l^{\circ}(t)\}$, and also relations (2.18) and (2.19), we obtain the inequality

$$\Phi(\lambda^{\circ},\ l^{\circ}(t)) \leqslant \liminf_{N\to\infty} \Phi(\lambda_N^{\circ},\ l_N^{\circ}(t)) + \sigma/2$$

This inequality clearly contradicts our assumption.

Thus, there exists a subsequence $\{\xi_N{}^{\circ}\}$ of quantities which ensures simultaneous fulfillment of conditions (2.14) and (2.15). Taking the limit, we obtain the quantity $\xi^{\circ} = \{\lambda^{\circ}, l^{\circ}(t)\}$. Let us consider the subsequence $\{\nu_N{}^{\circ}\}$ of numbers corresponding to this subsequence $\{\xi_N{}^{\circ}\}$. Relations (2.6) imply the inequality $\nu_{N_2}{}^{\circ} \geq \nu_{N_1}{}^{\circ}$ for all $N_2 > N_1$.

The subsequence $\{\nu_N{}^{\circ}\}$ is therefore monotonous; it is bounded and converges to the finite limit ν°. Taking the limit $(N \to \infty)$ in (2.9) and (2.10), we obtain Eq.

$$\nu^{\circ} = \Psi(\xi^{\circ}) = \{\varphi_1[\xi^{\circ}] - \varphi_3[l^{\circ}(t)]\} \left(\int_{t_0}^{T} |\varphi_2[\xi^{\circ};\ \tau]|\,d\tau\right)^{-1} \tag{2.20}$$

Assuming the contrary and making use of representations (2.18) and (2.19), we conclude that the following condition holds:

$$\nu^{\circ} = \Psi(\xi^{\circ}) = \max_{\xi} \Psi(\xi) = \max_{\lambda,\, l(t)} \{\varphi_1[\xi] - \varphi_3[l(t)]\} \left(\int_{t_0}^{T} |\varphi_2[\xi;\ \tau]|\,d\tau\right)^{-1} \tag{2.21}$$

for

$$\rho[\xi] = \rho[\lambda,\quad l(t)] \leqslant 1, \quad \operatorname{vrai\,max}_t \sum_{k=1}^{m} |l_k(t)| \leqslant 1$$

The quantity $\xi^{\circ} = \{\lambda^{\circ}, l^{\circ}\}$ thus turns out to be an extremal element of arbitrary extremum problem (2.21), which is the limiting case for problem (2.6), (2.7).

Let us show that from the sequence of optimal controls $u_N{}^{\circ}(t)$ (2.4) for Problems 1_N we can isolate a subsequence having the weak limit $u^{\circ}(t)$, and that his limit $u^{\circ}(t)$ is, in fact, the optimal control for Problem 1. (Under the indicated conditions the sequence of trajectories $x[t;\, u_N{}^{\circ}]$ converges uniformly to the optimal trajectory $x[t;\, u^{\circ}]$.)

In fact, the quantities $u_N{}^{\circ}(t) = \nu_N{}^{\circ}\,\mathrm{sign}\, h_N{}^{\circ}(t)$ are bounded in the metric of L_2 and therefore contain the subsequence $\{u_N{}^{\circ}\}$ which converges weakly in L_2 to some function $u^{\circ}(t)$. The function $u^{\circ}(t)$ clearly satisfies the conditions of Problem 1 (see (2.2) and (2.3)). Here we have [2]: $\operatorname{vrai\,max}_t |u^{\circ}(t)| \leq \nu^{\circ}$. We shall show that $\operatorname{vrai\,max}_t |u^{\circ}(t)| = \nu^{\circ}$. In fact, assuming that $\operatorname{vrai\,max}_t |u^{\circ}(t)| = \eta < \nu^{\circ}$, we can find a number N such that $\eta < \nu_N{}^{\circ} =$

$= \text{vrai max}_t |u_N^\circ(t)| < \nu^\circ$. The latter contradicts the optimality of the control u_N°. By similar reasoning we can show that the control $u^\circ(t)$ is optimal.

Let us describe briefly the structure of the function $u^\circ(t)$. First, we exclude the case where the neighborhood of each point of the set $[t_0, T]$, where $h^\circ(t) = 0$, can contain points from $[t_0, T]$, where $h^\circ(t) \neq 0$. We begin by considering the set $e \subset [t_0, T]$, where $h^\circ(t) > 0$. The set e is open.

Let us choose a sequence $\{\gamma_k\}$ of positive numbers γ_k which converges to zero. By e_k we denote the set $e_k \subset e$, where $h^\circ(t) \geq \gamma_k$. We choose a number $k = j$ such that the set e_j is nonempty. The set e_j is closed. Making use of (2.8), we represent the functions $h_N^\circ(t)$ (2.5) in the form $h_N^\circ(\tau) = \phi_2[\xi_N^\circ; \tau] + o_2(\Delta_N t)$. These functions are generally not continuous. On the other hand, the functions $\phi_2[\xi_N^\circ; \tau]$ (2.12) are continuous and (by virtue of condition (2.10) and the properties of the quantities $h_*[T, \tau]$, $h_k[t, \tau]$) form a set compact [2] in the space C. Hence, there exists a subsequence $\{\xi_N^\circ\}$ of quantities (we use our original symbol to denote this subsequence) on which the convergence of the functions $\phi_2[\xi_N^\circ; \tau]$ to the function $h^\circ(\tau)$ (2.17) is uniform.

Choosing the numbers N_1 and N_2 in such a way that $\phi_2[\xi_N; \tau] \geq 2\gamma_j/3$ for $N > N_1$ and $o_2(\Delta_N t) \leq \gamma_j/3$ for $N > N_2$, we see that $h_N^\circ(t) \geq \gamma_j/3$ for $N = N(j) = \max(N_1, N_2)$. In accordance with (2.4), we find that the subsequence of controls $u_N^\circ(t)$ converges on the set e_j to the constant ν°.

It follows from this that the weak limit $u^\circ(t)$ is also equal to ν° on the set e_j. Reasoning in this way for each $k > j$, we see that $u^\circ(t) = \nu^\circ$ on each of the corresponding sets e_k. Further, recalling that $e = \bigcup e_k$, we find that $u^\circ(t) = \nu^\circ$ if $h^\circ(t) > 0$. Similarly, we can show that $u^\circ(t) = -\nu^\circ$ if $h^\circ(t) < 0$. Thus, we conclude that $u^\circ(t) = \nu^\circ \operatorname{sign} h^\circ(t)$ if $h^\circ(t) \not\equiv 0$ and that $u^\circ(t)$ is the weak limit of the subsequence of functions $u_N^\circ(t)$ if $h^\circ(t) \equiv 0$.

The above implies that the optimal control $u^\circ(t)$ satisfies the following maximum relation:

$$h^\circ(t)\, u^\circ(t) = \max_u h^\circ(t)\, u(t) \quad \text{for} \quad \text{vrai max}_t\, |u(t)| \leqslant \nu^\circ \tag{2.22}$$

Expression (2.22) for Problem 1 is analogous to the Pontriagin maximum principle and is similar to the necessary conditions of optimality of the control u° obtained for problems of this type in [5].

We note, however, that the limiting process under consideration establishes the existence of the solution of the problem, provides additional condition (2.21) which defines the function $h^\circ(t)$, and yields the value of ν° which serves as an estimate of the optimal control u°. Finally (and this is the most important result of our investigation), the limiting process enables us to find the optimal control in those time intervals where $h^\circ(t) \equiv 0$.

In fact, Condition (2.22) does not tell us how to choose the control $u^\circ(t)$ when $h^\circ(t) \equiv 0$.

We noted above that the optimal control $u^\circ(t)$ can be sought in this case as the weak limit of a subsequence of functions $u_N^\circ(t)$.

However, actual computation of $u^\circ(t)$ by this method is made difficult by the fact that the control $u_N^\circ(t)$ in time intervals when $h^\circ(t) \equiv 0$ takes the form of discontinuous controls with the number of switchings increasing as $N \to \infty$. This leads to a so-called "sliding state" in system (1.1). In order to circumvent this difficult at least partially, let us consider the following method of constructing $u^\circ(t)$ in time intervals when $h^\circ(t) \equiv 0$. Let t be an arbitrary point in one such interval, and let us consider the functions

$$u_\varepsilon^\circ(t) = \lim_{N\to\infty} u_{\varepsilon N}^\circ(t) = \lim_{N\to\infty} \frac{1}{\varepsilon} \int_0^\varepsilon u_N^\circ(t+\vartheta)\, d\vartheta \tag{2.23}$$

Functions (2.23) are continuous [2]. The essence of operation (2.23) lies in the fact that the subsequence $\{u_N^\circ\}$ of generally discontinuous controls u_N° (2.4) is replaced here by the sequence $\{u_{\varepsilon_k}^\circ\}$ ($\varepsilon_k \to 0$ as $k \to \infty$) average (continuous) controls u_ε° (2.23), so that we can speak of a weak limit of $\{u_{\varepsilon_k}^\circ(t)\}$ which we shall call the "regularized optimal control". We shall merely verify here that $\{u_{\varepsilon_k}^\circ(t)\}$ yields the same trajectory as the control

$u^{\circ}(t)$. Constructing the differences $|x_s[t; u_\varepsilon] - x_s[t, u^{\circ}]|$ and recalling the weak convergence of $\{u_N^{\circ}\}$ to u°, we obtain

$$|x_s[t, u_\varepsilon^0] - x_s[t, u^\circ]| = \left|\frac{1}{\varepsilon}\int_0^\varepsilon\left\{\lim_{N\to\infty}\left[\int_\vartheta^{t+\vartheta} h_s[t, \eta-\vartheta]\, u_{\varepsilon N}^\circ(\eta)\, d\eta - \right.\right.\right.$$

$$\left.\left.\left. - \int_0^t h_s[t, \eta]\, u^\circ(\eta)\, d\eta\right]\right\} d\vartheta\right| \leqslant \frac{1}{\varepsilon}\int_0^\varepsilon\left\{\left|\int_\vartheta^{t+\vartheta} [h_s[t, \eta-\vartheta] - h_s[t, \eta]\, u^\circ(\eta)\, d\eta\right| + \right.$$

$$+ \left|\int_\vartheta^t [h_s[t, \eta-\vartheta] - h_s[t, \eta]]\, u^\circ(\eta)\, d\eta\right| +$$

$$\left. + \left|\int_0^\vartheta [h_s[t, \eta-\vartheta] - h_s[t, \eta]]\, u^\circ(\eta)\, d\eta\right|\right\} d\vartheta \leqslant \varepsilon$$

Thus, we find that the condition

$$\lim_{\varepsilon\to 0} x_s[t, u_\varepsilon^\circ] = x_s[t, u^\circ] \qquad (s = 1, \ldots, n)$$

is fulfilled for all $t_0 \le t \le T$.

The foregoing is summarized by the following.

T h e o r e m. The control $u^{\circ}(t)$ obtained as the weak or regularized limit of the optimal controls u_N° in Problems 1_N is optimal for Problem 1. It satisfies maximum principle (2.22) where the minimal function $h^{\circ}(t)$ and the number ν° are the solution of arbitrary extremum problem (2.21), which is the limiting case of problem (2.6), (2.7). In those intervals where $h_0(t) \not\equiv 0$ the control $u^{\circ}(t) = \nu^{\circ}$ sign $h^{\circ}(t)$. In those intervals where $h^{\circ}(t) \equiv 0$ the control $u^{\circ}(t)$ can be found by taking the regularized limit of the continuous functions $u_\varepsilon^{\circ}(t)$ (2.23) as $\varepsilon \to 0$.

In the same way we can solve the problem of time-optimal operation with specified restrictions on the control ($\varkappa[u] \le \nu$) and on the phase coordinates (1.2). The difference lies in the fact that in problems (2.6), (2.7) and (2.21) the unknown is the time T, while the constant ν is given (*).

N o t e s. 1. Eqs. (2.23) represent just one of the methods of constructing the control $u^{\circ}(t)$ for $h^{\circ}(t) \equiv 0$. Other methods of constructing $u^{\circ}(t)$ in such cases will be investigated in a later paper.

2. The above arguments remain valid for Problem 1 with norms $\varkappa[u]$ of forms other than $\varkappa[u] = \text{vrai max}_t |u(t)|$. They can be generalized automatically for the case of a convex positive functional $\varkappa[u]$ and for a nonsteady-state System (1.1) with the vector control u.

3. The above approach to the solution of Problem 1 also covers problems on the minimum in a given time interval $t_0 \le t \le T$ of the maximal deviation of the phase coordinates of System (1.1) under the specified restriction $\varkappa[u] \le \nu$.

As an elementary example (which nevertheless affords a clear notion of all the basic operations at the basis of the described method, and which can readily be solved on the basis of simple mechanical considerations), let us consider the motion $x^{\cdot\cdot} = u$ of a material point which must be transferred by means of the force u ($|u| \le 1$) in the minimal time T from the position $\{x(0) = 0,\ x^{\cdot}(0) = 0\}$ to the position $\{x(T) = 1,\ x^{\cdot}(T) = 0\}$ under the restriction $|x^{\cdot}(t)| \le f(t) = t/2$. Problem (2.21) reduces to that of finding

$$\max_{\lambda,\ l(t)}\left\{\left(\lambda_2 - \int_0^T 0.5t\,|\,l(t)\,|\,dt\right)\left(\int_0^T \left|(T-\tau)\lambda_1 + \lambda_2 + \int_\tau^T l(t)\,dt\right| d\tau\right)^{-1}\right\} = 1$$

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*) A particular case of this problem (without discussion of the case $h^{\circ}(t) \equiv 0$) is considered in [6].

for

$$\lambda_1^2 + \lambda_2^2 + \int_0^T l(t)\,dt = r, \qquad |l(t)| \leqslant r, \qquad r = \frac{5 + 2\sqrt{6}}{3}$$

(The arguments in the first half of the present paper remain valid even if the restrictions in (2.7) are not unitary.)

Solution of this problem yields the values

$$\lambda_1^\circ = 1, \qquad \lambda_2^\circ = -\sqrt{6}/3, \qquad T = \sqrt{6}$$

$$l^\circ(t) = -1 \quad \text{for} \quad 0 \leqslant t \leqslant 2\sqrt{6}/3, \qquad l^\circ(t) \equiv 0 \quad \text{for} \quad 2\sqrt{6}/3 < t \leqslant \sqrt{6}$$

From (2.17) we find that

$$h^\circ(t) \equiv 0 \quad \text{for} \quad 0 \leqslant t \leqslant 2\sqrt{6}/3, \qquad h^\circ(t) = -t + 2\sqrt{6}/3 \quad \text{for} \quad 2\sqrt{6}/3 \leqslant t \leqslant \sqrt{6}$$

According to our Theorem, $u^\circ(t) = -1$ for $2\sqrt{6}/3 \leq t \leq \sqrt{6}$. We construct the functions $u_\varepsilon^\circ(t)$ (2.23) for determining $u^\circ(t)$ for $0 \leq t < 2\sqrt{6}/3$. To this end, solving Problem 1_N with the interval $\Delta_N t = 2\sqrt{6}/3N$, we find that when $0 \leq t \leq 2\sqrt{6}/3$,

$$u^\circ(t) = 1 \quad \text{for} \quad t_{i-1} \leqslant t < t_i + \Delta_N t\,{}^3/_4$$

$$u^\circ(t) = 1 \quad \text{for} \quad t_{i-1} + 3\Delta_N t/4 \leqslant t < t_i + \Delta_N t$$

(As we see, the controls $u_N^\circ(t)$ produce a "sliding state".) The regularized functions $u_\varepsilon^\circ(t)$ (2.23) are of the form $u_\varepsilon^\circ = 1/2$ $(0 \leq t < 2\sqrt{6}/3 - \varepsilon)$ and yield the regularized sequence $\{u^\circ\}$ of controls which converges (already in the usual sense in our example) to the function $u^\circ(t) = 1/2$, which is the weak limit of the quantities $u_N^\circ(t)$. Finally,

$$u^\circ(t) = {}^1/_2 \quad \text{for} \quad 0 \leqslant t < 2\sqrt{6}/3$$

$$\text{and } u^\circ(t) = -1 \quad \text{for} \quad 2\sqrt{6}/3 \leqslant t \leqslant \sqrt{6}.$$

The problem considered in the present paper was formulated in N.N. Krasovskii's seminar at Ural University. The authors are grateful to Prof. Krasovskii for his valuable suggestions.

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Translated by A.Y.